Zoom Class 23/04/2020

Polar form problems

Ques Find the radius of curvature at any point for the curve $r = ae^{m\theta}$

Soln $\because r = ae^{m\theta}$ is in polar form

$$r_1 = \frac{dr}{d\theta} = ae^{m\theta} \cdot m = m \cdot r \quad \text{——} \ (1)$$

Diffrentiating (1) w.r.t. θ we get

$$r_2 = \frac{d^2 r}{d\theta^2} = m \cdot \frac{dr}{d\theta} = m(mr) = m^2 r \quad \text{——} \ (2)$$

Now, since $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$

$$= \frac{(r^2 + m^2 r^2)^{3/2}}{r^2 + 2m^2 r^2 - m^2 r^2}$$

$$= \frac{(r^2 + m^2 r^2)^{3/2}}{(r^2 + m^2 r^2)} = \frac{(r^2)^{3/2} \{1 + m^2\}^{3/2}}{r^2 \{1 + m^2\}}$$

$$= \frac{r^3}{r^2} \cdot (1 + m^2)^{3/2 - 1}$$

$$= r \cdot (1 + m^2)^{1/2}$$

$$= r\sqrt{1 + m^2}$$

Ans

Ques Find the radius of curvature at any point of the curve $r = a(1+\cos\theta)$

Soln Given $r = a(1+\cos\theta)$

Differentiating w.r.t. θ, we get

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta \quad \text{——— (1)}$$

and $$r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta \quad \text{——— (2)}$$

$$\because \rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 + rr_2}$$

Using (1) and (2) we get

$$\rho = \frac{(r^2 + (-a\sin\theta)^2)^{3/2}}{r^2 + 2(-a\sin\theta)^2 + r(-a\cos\theta)}$$

$$\Rightarrow \rho = \frac{\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}^{3/2}}{\{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a\cos\theta\cdot(1+\cos\theta)a}$$

$$= \frac{\{a^2 + 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta\}^{3/2}}{\{a^2(1+2\cos\theta+\cos^2\theta) + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta\}}$$

$$= \frac{a^3\{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\}^{3/2}}{a^2\{1+2\cos\theta+\cos^2\theta + 2\sin^2\theta + (\cos\theta + \cos^2\theta)\}}$$

$$= \frac{a\{1 + 2\cos\theta + 1\}}{\{1+3\cos\theta + 2(\cos^2\theta + \sin^2\theta)\}}$$

$$\Rightarrow \rho = \frac{a\{2+2\cos\theta\}^{3/2}}{\{1+3\cos\theta+2\}}$$

$$= \frac{2^{3/2} a \cdot \{1+\cos\theta\}^{3/2}}{\{3+3\cos\theta\}} = \frac{2^{3/2} \cdot a\{1+\cos\theta\}^{3/2}}{3\{1+\cos\theta\}}$$

$$= \frac{2^{3/2}}{3} \cdot a\{1+\cos\theta\}^{\frac{3}{2}-1}$$

$$= \frac{2^{3/2}}{3} \cdot a\{1+\cos\theta\}^{1/2}$$

$$= \frac{2^{3/2}}{3} \cdot a\{2\cos^2(\theta/2)\}^{1/2}$$

$$= \frac{2^{3/2} \cdot 2^{1/2}}{3} \cdot a\{\cos^2\theta/2\}^{(1/2)}$$

$$= \frac{2^2}{3} \cdot a \cdot \cos\left(\frac{\theta}{2}\right)$$

$$= \frac{4}{3} a \cos\frac{\theta}{2}$$

Ans

Ques If $r = a\sec 2\theta$, Prove that $\rho = \frac{r^4}{3p^3}$

Soln Since the required proof is in pedal form we convert given eqn in pedal form

i.e $r = a\sec 2\theta$

$\Rightarrow \log r = \log(a \cdot \sec 2\theta)$

$\Rightarrow \log r = \log a + \log \sec 2\theta$ —— ①

Diffrentiating ① w.r.t. θ we get

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sec 2\theta} \times (\sec 2\theta \tan 2\theta) 2$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 2\tan 2\theta$$

$$\Rightarrow r\frac{d\theta}{dr} = 1/2\tan 2\theta$$

$$\Rightarrow \tan\phi = \frac{1}{2\tan 2\theta} \quad \text{—— ①}$$

We know that

$$p = r\sin\phi$$

$$= r \cdot \frac{1}{\sqrt{1 + 4\tan^2 2\theta}} \quad \text{—} \left\{ \because \sin\phi = \frac{1}{\sqrt{1+\cot^2\phi}} \text{ and } \cot\phi = 2\tan 2\theta \text{ from Eqn ①} \right\}$$

$$\Rightarrow p = r\cdot\frac{1}{\sqrt{4(\sec^2 2\theta-1)+1}} = r\cdot\frac{1}{\sqrt{4\sec^2 2\theta-4+1}}$$

$$= r\cdot\frac{1}{\sqrt{4\sec^2 2\theta-3}}$$

$$\cancel{\Rightarrow p^2 = \frac{r^2}{4\sec^2 2\theta-3}} \Rightarrow p^2 = \frac{r^2}{4\sec^2 2\theta-3} \quad \text{—— (2)}$$

Also since $r = a\sec 2\theta \Rightarrow \sec 2\theta = \frac{r}{a}$
therefor (2) implies

$$\Rightarrow p^2 = \frac{r^2}{4\left(\frac{r}{a}\right)^2-3} = \frac{r^2}{(4r^2-3a^2)/a^2}$$

$$= \frac{r^2\cdot a^2}{4r^2-3a^2}$$

$$\Rightarrow 4r^2p^2-3a^2p^2 = r^2a^2 \quad \text{—— (A)}$$

$$\Rightarrow 4r^2p^2-r^2a^2 = 3a^2p^2 \Rightarrow r^2(4p^2-a^2) = 3a^2p^2 \quad \text{—— (3)}$$

Diffrentiating (3) w.r.t. p we get

$$\Rightarrow r^2(8p)+(4p^2-a^2)2r\frac{dr}{dp} = 3a^2\cdot 2p$$

$$\Rightarrow \frac{8r^2p}{2} + \frac{(4p^2-a^2)\cancel{2}r}{\cancel{2}}\cdot\frac{dr}{dp} = 3a^2p$$

$$\Rightarrow 4r^2p + (4p^2-a^2)\rho = 3a^2p$$

$$\cancel{\Rightarrow (4p^2-a^2)\rho = \frac{3a^2 r}{4r^2p}} \quad \cancel{\Rightarrow \rho = 3a}$$

$$\Rightarrow (4p^2-a^2)\rho = p(3a^2-4r^2)$$

$$\Rightarrow \frac{3a^2p^2}{r^2}\cdot\rho = \cancel{p}\left(-\frac{r^2a^2}{p^{\cancel{2}}}\right) \quad \text{— from Eqn (A) and (B)}$$

$$\Rightarrow \rho = -\frac{r^2a^2}{p}\times\frac{r^2}{3a^2p^2}$$

$$\Rightarrow \rho = -\frac{1}{3}\frac{r^4}{p^3}$$

Ans